

Existence of solutions for a semirelativistic Hartree equation with unbounded potentials

Simone Secchi *

January 12, 2017

To Francesca, always

We prove the existence of a solution to the semirelativistic Hartree equation

$$\sqrt{-\Delta + m^2}u + V(x)u = A(x) (W * |u|^p) |u|^{p-2}u$$

under suitable growth assumption on the potential functions V and A . In particular, both can be unbounded from above.

1 Introduction

The mean field limit of a quantum system describing many self-gravitating, relativistic bosons with rest mass $m > 0$ leads to the time-dependent pseudo-relativistic Hartree equation

$$i \frac{\partial \psi}{\partial t} = \left(\sqrt{-\Delta + m^2} - m \right) \psi - \left(\frac{1}{|x|} * |\psi|^2 \right) \psi, \quad x \in \mathbb{R}^3 \quad (1.1)$$

where $\psi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$ is the wave field. Such a physical system is often referred to as a boson star in astrophysics (see [16, 20, 21]). Solitary wave solutions $\psi(t, x) = e^{-it\lambda} \phi$, $\lambda \in \mathbb{R}$ to equation (1.1) satisfy the equation

$$\left(\sqrt{-\Delta + m^2} - m \right) \phi - \left(\frac{1}{|x|} * |\phi|^2 \right) \phi = \lambda \phi \quad \text{in } \mathbb{R}^3.$$

For the non-relativistic Hartree equation driven by the *local* differential operator $-\Delta + m^2$, existence and uniqueness (modulo translations) of a minimizer were proved by Lieb [22] by using symmetric decreasing rearrangement inequalities. Within the same setting, always for the negative Laplacian, P.-L. Lions [25] proved existence of infinitely many

*The author is supported by the MIUR 2015 PRIN project “Variational methods, with applications to problems in mathematical physics and geometry”.

spherically symmetric solutions by application of abstract critical point theory both with and without constraints for a more general radially symmetric convolution potential. The non-relativistic Hartree equations is also known as the Choquard-Pekard or Schrödinger-Newton equation and recently a large amount of papers are devoted to the study of solitary states and its semiclassical limit: see [1, 4, 5, 7, 8, 9, 11, 12, 13, 14, 20, 23, 27, 26, 29, 30, 31, 38, 39] and references therein.

In this paper, which is somehow a continuation of the investigation we began in [33, 34], we consider the equation

$$\sqrt{-\Delta + m^2}u + V(x)u = A(x) (W * |u|^p) |u|^{p-2}u \quad (1.2)$$

in the weighted space

$$X = \left\{ u \in L^2(\mathbb{R}^N) \mid \|u\|_X < \infty \right\},$$

where

$$\|u\|_X^2 = \int_{\mathbb{R}^N} \left| (-\Delta + m^2)^{1/4} u \right|^2 dx + \int_{\mathbb{R}^N} V(x) |u|^2 dx.$$

In the case $A \equiv 1$, equation (1.2) has been recently investigated in [12], where least-energy solutions are constructed for a bounded potential V .

We collect our assumptions.

(H1) $N > 1$, $\beta > 1$, $2 \leq p < \frac{2N\beta}{\beta(N-1)}$ and $r > \max \left\{ 1, \frac{1-N\beta}{2+\beta N(p-2)-\beta p} \right\}$.

(H2) $A \in L_{\text{loc}}^\infty(\mathbb{R}^N)$ satisfies $A(x) \geq 1$ for almost every $x \in \mathbb{R}^N$.

(H3) $V: \mathbb{R}^N \rightarrow \mathbb{R}$ is continuous and

$$\inf_{\mathbb{R}^N} V > -B$$

for some $B > 0$; furthermore,

$$\lambda_1 = \inf_{u \in X \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \left| (-\Delta + m^2)^{1/4} u \right|^2 + \int_{\mathbb{R}^N} V |u|^2}{\|u\|_2^2} > 0.$$

(H4) There exist $C_0 > 0$ and $R_0 > 0$ such that

$$A(x)^{\frac{2r}{2r-1}} \leq C_0 \left(1 + (\max\{0, V(x)\})^{\frac{1}{\beta}} \right)$$

for all $|x| > R_0$.

(H5) $W: \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}$ and $W = W_1 + W_2$, where $W_1 \in L^r$ and $W_2 \in L^\infty$.

Equivalently, condition (H4) can be stated as

$$\limsup_{|x| \rightarrow +\infty} \frac{A(x)^{\frac{2r}{2r-1}}}{1 + (\max\{0, V(x)\})^{\frac{1}{\beta}}} < +\infty.$$

Remark 1.1. We highlight that both the potential V and the potential A in front of the nonlinear term in the right-hand side can be unbounded. To the best of our knowledge, this case is treated here for the first time. In the very recent paper [6], a similar equation has been studied under an assumption that, in our framework, reads as $V \equiv 1$ and $\lim_{|x| \rightarrow +\infty} A(x) = 1$.

To state our last assumption, we define, for any open subset $\Omega \subset \mathbb{R}^N$ and any $t \geq 2$, the quantity (see Section 2 for the definition of the space $L^{1/2,2}(\Omega)$)

$$\begin{aligned} \nu(t, \Omega) &= \inf_{\substack{u \in L^{1/2,2}(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} |(-\Delta + m^2)^{1/4} u|^2 + \int_{\Omega} V |u|^2}{(\int_{\Omega} |u|^t)^{2/t}} \quad \text{if } \Omega \neq \emptyset \\ &= +\infty \quad \text{if } \Omega = \emptyset. \end{aligned}$$

We assume that

(ν) there exists $t_0 \in [2, 2N(N-1))$ such that

$$\lim_{R \rightarrow +\infty} \nu(t_0, \mathbb{R}^N \setminus \overline{B_R}) = +\infty,$$

where $B_R = \{x \in \mathbb{R}^N \mid |x| < R\}$.

The number λ_1 in assumption (H3) can be actually seen as an *eigenvalue*, as the following result shows.

Proposition 1.2. *Assume that $V: \mathbb{R}^N \rightarrow \mathbb{R}$ is continuous and $\inf_{\mathbb{R}^N} V > -B$ for some $B > 0$. Assume also that (ν) holds with $t_0 = 2$. Then there exists $\varphi_1 \in X$ such that*

$$\sqrt{-\Delta + m^2} \varphi_1 + V(x) \varphi_1 = \lambda_1 \varphi_1 \quad \text{in } \mathbb{R}^N.$$

Furthermore φ_1 can be assumed to be positive in \mathbb{R}^N .

Proof. It follows immediately from the assumptions that $\lambda_1 > -B > 0$. Let $\{u_n\}_n$ be a minimizing sequence for λ_1 , in the sense that

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \left| (-\Delta + m^2)^{\frac{1}{4}} u_n \right|^2 + \int_{\mathbb{R}^N} V |u_n|^2 = \lambda_1 \tag{1.3}$$

$$\int_{\mathbb{R}^N} |u_n|^2 = 1. \tag{1.4}$$

As in [10, Lemma 2.1], we can assume that u_n is non-negative. It follows from (1.3) that

$$\int_{\mathbb{R}^N} \left| (-\Delta + m^2)^{\frac{1}{4}} u_n \right|^2 \leq \lambda_1 + 1 + B.$$

Let u be the weak limit of $\{u_n\}_n$ in $L^{1/2,2}(\mathbb{R}^N)$. Since $u_n \rightarrow u$ strongly in $L^2(B_R)$ for any $R > 0$, we may assume that u_n converges to u almost everywhere, and consequently

$u \geq 0$. We fix a smooth cut-off function ψ such that $\psi \equiv 0$ on B_R and $\psi \equiv 1$ on $\mathbb{R}^N \setminus B_{R+1}$. Then

$$\begin{aligned} \|u_n - u\|_2^2 &\leq \|(1 - \psi)(u_n - u)\|_2^2 + \|\psi(u_n - u)\|_2^2 \\ &\leq o_n(1) + \frac{1}{\nu(2, \mathbb{R}^N \setminus B_R)} \left(\int_{\mathbb{R}^N} \left| (-\Delta + m^2)^{\frac{1}{4}} (u_n - u) \right|^2 + \int_{\mathbb{R}^N} V |u_n - u|^2 \right) \\ &\leq o_n(1) + o_R(1), \end{aligned}$$

where $\lim_{n \rightarrow +\infty} o_n(1) = 0$ and $\lim_{R \rightarrow +\infty} o_R(1) = 0$: this proves that $u_n \rightarrow u$ strongly in $L^2(\mathbb{R}^N)$. In particular, $\int_{\mathbb{R}^N} |u|^2 = 1$ due to (1.4). Plainly,

$$\int_{\mathbb{R}^N} \left| (-\Delta + m^2)^{\frac{1}{4}} u \right|^2 \leq \liminf_{n \rightarrow +\infty} \left| (-\Delta + m^2)^{\frac{1}{4}} u_n \right|^2.$$

Set $G = \{x \in \mathbb{R}^N \mid V(x) > 1\}$. Since $V \in L^\infty(\mathbb{R}^N \setminus G)$, we have

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N \setminus G} V |u_n|^2 = \int_{\mathbb{R}^N \setminus G} V |u|^2.$$

On the other hand, by [19, Theorem 6.54] we deduce that

$$v \in L^2(G) \mapsto \int_G V |v|^2$$

is weakly lower semicontinuous. To summarize,

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^N} V |u_n|^2 &= \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N \setminus G} V |u_n|^2 + \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^N} V |u_n|^2 \\ &\geq \int_{\mathbb{R}^N} V |u|^2. \end{aligned}$$

Hence u is a minimizer for λ_1 . The strict positivity of u follows from [17, Proposition 2]. \square

Remark 1.3. In the previous Proposition, we did *not* assume that $\lambda_1 > 0$. Under this additional assumption, the proof would be much easier.

It is important to remark that condition (ν) is related to some other popular conditions. In Proposition 3.2 we prove that the coercivity assumption

$$\lim_{|x| \rightarrow +\infty} V(x) = +\infty$$

implies (ν) . Similarly, the condition

(ν') For every $b > 0$, the set V^b has finite Lebesgue measure

also implies (ν) , see Proposition 3.4. Finally, Sirakov's condition [35]

(ν'') There exists $t_1 \in [2, 2N/(N-1))$ such that, for any $r > 0$ and any sequence $\{x_n\}_n$ of points in \mathbb{R}^N such that $\lim_{n \rightarrow +\infty} |x_n| = +\infty$, there results $\lim_{n \rightarrow +\infty} \nu(t_1, B(x_n, r)) = +\infty$

is equivalent to (ν): see Proposition 3.5.

Remark 1.4. While dealing with the semirelativistic Hartree equation, it is customary to rewrite the operator $\sqrt{-\Delta + m^2} + V$ as $\sqrt{-\Delta + m^2} - m + (V + m)$ in order to exploit the fact that $\sqrt{-\Delta + m^2} - m > 0$ in the sense of functional calculus. The natural assumption from a variational point of view is therefore $\inf_{\mathbb{R}^N} V > -m$. Our assumption (H3) is, in general, less restrictive as it only requires a *suitable* lower bound for V .

We can state the main result of this paper.

Theorem 1.5. *Suppose that (H1), (H2), (H3), (H4), (H5), and (ν) are satisfied. Then equation (1.2) has infinitely many distinct solutions.*

The proof is based on the ideas developed in [35] for a local equation and a pointwise nonlinearity.

Remark 1.6. It should be noticed that our results continue to hold if we replace (1.2) with

$$(-\Delta + m^2)^s u + V(x)u = (W * |u|^p) |u|^{p-2}u$$

with $0 < s < 1$ and $N > 2s$, for instance $N \geq 3$. Of course some numbers must be replaced: $2N/(N-1)$ should become $2N/(N-2s)$, and so on. We prefer to work out the details for $s = 1/2$, which corresponds to the physical model of the Hartree equation.

In Section 2 we introduce the necessary preliminaries on function space. In Section 3 we compare several assumptions that ensure a compact embedding result. In Section 4 we prove a compactness theorem that is used in Section 5 to prove our main existence result.

Notation

1. The letters c and C will stand for a generic positive constant that may vary from line to line.
2. The operator d will be reserved for the (Fréchet) derivative, so that dI denotes the Fréchet derivative of a function I .
3. The symbol \mathcal{L}^N will be reserved for the Lebesgue N -dimensional measure.
4. The Fourier transform of a function u will be denoted by $\mathcal{F}u$.

2 Variational setting

Let us recall the definition of the *Bessel function space* defined for $\alpha > 0$ by

$$L^{\alpha,2}(\mathbb{R}^N) = \left\{ f : f = G_\alpha \star g \text{ for some } g \in L^2(\mathbb{R}^N) \right\},$$

where the Bessel convolution kernel is defined by

$$G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2} \Gamma(\alpha/2)} \int_0^\infty \exp\left(-\frac{\pi}{t}|x|^2\right) \exp\left(-\frac{t}{4\pi}\right) t^{\frac{\alpha-N}{2}-1} dt$$

The norm of this Bessel space is $\|f\| = \|g\|_2$ if $f = G_\alpha \star g$. The operator

$$(I - \Delta)^{-\alpha} u = G_{2\alpha} \star u$$

is usually called Bessel operator of order α . In Fourier variables the same operator reads

$$G_\alpha = \mathcal{F}^{-1} \circ \left((1 + |\xi|^2)^{-\alpha/2} \circ \mathcal{F} \right),$$

so that

$$\|f\| = \left\| (I - \Delta)^{\alpha/2} f \right\|_2.$$

For more detailed information, see [2, 36] and the references therein. The use of $(-\Delta + m^2)^{\alpha/2}$ instead of $(-\Delta + I)^{\alpha/2}$ is clearly harmless. We summarize the embedding properties of Bessel spaces. For the proofs we refer to [18, Theorem 3.2], [36, Chapter V, Section 3] and [37, Section 4].

Theorem 2.1. 1. $L^{\alpha,2}(\mathbb{R}^N) = W^{\alpha,2}(\mathbb{R}^N) = H^\alpha(\mathbb{R}^N)$.

2. If $\alpha \geq 0$ and $2 \leq q \leq 2_\alpha^* = 2N/(N - 2\alpha)$, then $L^{\alpha,2}(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$; if $2 \leq q < 2_\alpha^*$ then the embedding is locally compact.

3. Assume that $0 \leq \alpha \leq 2$ and $\alpha > N/2$. If $\alpha - N/2 > 1$ and $0 < \mu \leq \alpha - N/2 - 1$, then $L^{\alpha,2}(\mathbb{R}^N)$ is continuously embedded into $C^{1,\mu}(\mathbb{R}^N)$. If $\alpha - N/2 < 1$ and $0 < \mu \leq \alpha - N/2$, then $L^{\alpha,2}(\mathbb{R}^N)$ is continuously embedded into $C^{0,\mu}(\mathbb{R}^N)$.

Remark 2.2. Although the Bessel space $L^{\alpha,2}(\mathbb{R}^N)$ is topologically undistinguishable from the Sobolev fractional space $H^\alpha(\mathbb{R}^N)$, we will not systematically confuse them, since our equation involves the Bessel norm.

For a general open subset Ω of \mathbb{R}^N , the Bessel space $L^{\alpha,2}(\Omega)$ is defined as the space of the restrictions to Ω of functions in $L^{\alpha,2}(\mathbb{R}^N)$. In this paper we will always take $\alpha = 1/2$. As we said in the Introduction, we work in the weighted space

$$X = \left\{ u \in L^{1/2,2}(\mathbb{R}^N) \mid \|u\|_X < \infty \right\}$$

where

$$\|u\|_X^2 = \int_{\mathbb{R}^N} \left| (-\Delta + m^2)^{1/4} u \right|^2 dx + \int_{\mathbb{R}^N} V(x) |u|^2 dx$$

Lemma 2.3. *If assumption (H3) holds, then there exists a constant $\aleph > 0$ such that*

$$\|u\|_X \geq \aleph \|u\|_{L^{1/2,2}(\mathbb{R}^N)}$$

for every $u \in X$.

Proof. We proceed by contradiction. Assume that for some sequence $\{u_n\}_n$ in X there results

$$\|u_n\|_{L^{1/2,2}}^2 = 1, \quad \|u_n\|_X^2 < \frac{1}{n}.$$

By definition of $\lambda_1 > 0$, the last inequality entails $\|u_n\|_{L^2} = o(1)$. But then the contradiction

$$o(1) = -B\|u_n\|_{L^2}^2 \leq \int_{\mathbb{R}^N} V|u_n|^2 < \frac{1}{n} - 1$$

arises as $n \rightarrow +\infty$. \square

Solutions to (1.2) correspond to critical points of the functional $I: X \rightarrow \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\|u\|_X^2 - \frac{1}{2p} \int_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y)A(x)|u(x)|^p|u(y)|^p dx dy. \quad (2.1)$$

We need to prove that I is well-defined on X .

Proposition 2.4. *The space X is continuously embedded into the weighted Lebesgue space*

$$L^t(\mathbb{R}^N, A^{\frac{2r}{2r-1}} d\mathcal{L}^N)$$

for every $2 \leq t \leq \frac{2(N\beta-1)}{\beta(N-1)}$.

Proof. Let us decompose

$$\begin{aligned} \int_{\mathbb{R}^N} A(x)^{\frac{2r}{2r-1}} |u(x)|^t dx &= \int_{B(0,R_0)} A(x)^{\frac{2r}{2r-1}} |u(x)|^t dx \\ &\quad + \int_{\mathbb{R}^N \setminus B(0,R_0)} A(x)^{\frac{2r}{2r-1}} |u(x)|^t dx, \end{aligned} \quad (2.2)$$

where $R_0 > 0$ is the number defined in assumption (H4). Now,

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B(0,R_0)} A(x)^{\frac{2r}{2r-1}} |u(x)|^t dx &\leq C_0 \int_{\mathbb{R}^N \setminus B(0,R_0)} \left(1 + (\max\{0, V(x)\})^{\frac{1}{\beta}}\right) |u(x)|^t dx \\ &\leq C_0 \int_{\Omega_1} |u(x)|^t dx + C_0 \int_{\Omega_2} \left(1 + V(x)^{\frac{1}{\beta}}\right) |u(x)|^t dx, \end{aligned}$$

where

$$\Omega_1 = \{x \in \mathbb{R}^N \mid V(x) < 0\}, \quad \Omega_2 = \{x \in \mathbb{R}^N \mid V(x) \geq 0\}.$$

As a consequence, inserting this into (2.2),

$$\int_{\mathbb{R}^N} A(x)^{\frac{2r}{2r-1}} |u(x)|^t dx \leq 2C_0 \|u\|_{L^t}^t + \|A\|_{L^\infty(B(0,R_0))} \|u\|_{L^t}^t + \int_{\Omega_2} V(x)^{\frac{1}{\beta}} |u(x)|^t dx.$$

If we apply Hölder's inequality to the last integral, we obtain

$$\begin{aligned} \int_{\Omega_2} V(x)^{\frac{1}{\beta}} |u(x)|^t dx &= \int_{\Omega_2} V(x)^{\frac{1}{\beta}} |u(x)|^{\frac{2}{\beta}} |u(x)|^{t-\frac{2}{\beta}} dx \\ &\leq \left(\int_{\Omega_2} V(x) |u(x)|^2 dx \right)^{\frac{1}{\beta}} \left(\int_{\Omega_2} |u(x)|^{(t-\frac{2}{\beta}) \frac{\beta}{\beta-1}} dx \right)^{\frac{\beta-1}{\beta}}. \end{aligned}$$

But

$$\begin{aligned} \int_{\Omega_2} V(x) |u(x)|^2 dx &= \int_{\mathbb{R}^N} V(x) |u(x)|^2 dx - \int_{\{x \in \mathbb{R}^N \mid V(x) < 0\}} V(x) |u(x)|^2 dx \\ &\leq \|u\|_X^2 + B \int_{\mathbb{R}^N} |u(x)|^2 dx \leq \left(1 + \frac{B}{\lambda_1}\right) \|u\|_X^2. \end{aligned}$$

In conclusion,

$$\int_{\mathbb{R}^N} A(x)^{\frac{2r}{2r-1}} |u(x)|^t dx \leq C_1 \|u\|_{L^t}^t + C_2 \|u\|_X^{\frac{2}{\beta}} \|u\|_{L^{(t\beta-2)/(\beta-1)}}^{(t\beta-2)/\beta}. \quad (2.3)$$

It is elementary to check that

$$2 \leq \frac{t\beta - 2}{\beta - 1} \leq \frac{2N}{N - 1}$$

whenever

$$2 \leq t \leq \frac{2(N\beta - 1)}{\beta(N - 1)},$$

so that we can invoke the continuous embedding of X into $L^s(\mathbb{R}^N)$ for $2 \leq s \leq 2N/(N-1)$ and conclude from (2.3) that there exists a positive constant C such that

$$\int_{\mathbb{R}^N} A(x)^{\frac{2r}{2r-1}} |u(x)|^t dx \leq C \|u\|_X^t \quad \text{for every } u \in X.$$

This completes the proof. \square

Remark 2.5. Observe that when $\inf_{\mathbb{R}^N} V > 0$ and $A = 1$ identically, we can let $\beta \rightarrow +\infty$ and recover the weaker assumption $2 \leq t \leq 2N/(N-1)$.

To proceed further, we need the following inequality due to Hardy, Littlewood and Sobolev. We firstly recall that a function h belongs to the weak L^q space $L_w^q(\mathbb{R}^N)$ if there exists a constant $C > 0$ such that, for all $t > 0$,

$$\mathcal{L}^N \left(\left\{ x \in \mathbb{R}^N \mid |h(x)| > t \right\} \right) \leq \frac{C^q}{t^q}.$$

Its norm is then

$$\|h\|_{q,w} = \sup_{t>0} t \left(\mathcal{L}^N \left(\left\{ x \in \mathbb{R}^N \mid |h(x)| > t \right\} \right) \right)^{1/q}.$$

Proposition 2.6 ([24]). *Assume that p, q and t lie in $(1, +\infty)$ and $p^{-1} + q^{-1} + t^{-1} = 2$. Then, for some constant $N_{p,q,t} > 0$ and for any $f \in L^p(\mathbb{R}^N)$, $g \in L^t(\mathbb{R}^N)$ and $h \in L_w^q(\mathbb{R}^N)$, we have the inequality*

$$\left| \int_{\mathbb{R}^N \times \mathbb{R}^N} f(x)h(x-y)g(y) dx dy \right| \leq N_{p,q,t} \|f\|_p \|g\|_t \|h\|_{q,w}. \quad (2.4)$$

Writing $W = W_1 + W_2 \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ and using (2.4) we can estimate the convolution term as follows by means of Proposition 2.4:

$$\begin{aligned} \int_{\mathbb{R}^N} (W * |u|^p) A |u|^p &= \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x)|^p A(x) W(x-y) |u(y)|^p dx dy \\ &= \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x)|^p A(x) W_1(x-y) |u(y)|^p dx dy \\ &\quad + \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x)|^p A(x) W_2(x-y) |u(y)|^p dx dy \\ &\leq C \left(\int_{\mathbb{R}^N} |u|^{\frac{2rp}{2r-1}} A^{\frac{2r}{2r-1}} \right)^{\frac{2r-1}{2r}} \|W_1\|_r \left(\int_{\mathbb{R}^N} |u|^{\frac{2rp}{2r-1}} \right)^{\frac{2r-1}{2r}} \\ &\quad + \|W_2\|_\infty \left(\int_{\mathbb{R}^N} |u(x)|^p A(x) dx \right) \left(\int_{\mathbb{R}^N} |u(y)|^p dy \right) \\ &\leq C \left(\int_{\mathbb{R}^N} |u|^{\frac{2rp}{2r-1}} A^{\frac{2r}{2r-1}} \right)^{\frac{2r-1}{2r}} \|W_1\|_r \left(\int_{\mathbb{R}^N} A^{\frac{2r}{2r-1}} |u|^{\frac{2rp}{2r-1}} \right)^{\frac{2r-1}{2r}} \\ &\quad + \|W_2\|_\infty \left(\int_{\mathbb{R}^N} |u|^p A^{\frac{2r}{2r-1}} \right) \left(\int_{\mathbb{R}^N} |u|^p A^{\frac{2r}{2r-1}} \right) \\ &\leq C \|u\|_{L^{\frac{2rp}{2r-1}}(\mathbb{R}^N, A^{\frac{2r}{2r-1}} d\mathcal{L}^N)}^{2p} \|W_1\|_r + \|W_2\|_\infty \|u\|_{L^p(\mathbb{R}^N, A^{\frac{2r}{2r-1}} d\mathcal{L}^N)}^{2p}, \end{aligned} \quad (2.5)$$

where we have used several times the fact that $A(x) \geq 1$ for almost every $x \in \mathbb{R}^N$. Since

$$r > \max \left\{ 1, \frac{1 - N\beta}{2 + \beta N(p-2) - \beta p} \right\} \quad \text{and} \quad p < \frac{2N\beta}{\beta(N-1)},$$

we can use Proposition 2.4 and from (2.5) we see that the convolution term in I is finite. It is easy to check, by the same token and taking into account the assumption $p \geq 2$, that $I \in C^1(X)$.

Remark 2.7. As before, if $\inf_{\mathbb{R}^N} V > 0$ and $A = 1$ identically, we can recover the assumption

$$r > \max \left\{ 1, \frac{N}{p + (2-p)N} \right\}$$

used in [12].

3 Comparison between conditions on V

In this section we prove that condition (ν) is actually weaker than both the coercivity of V and of condition (ν') . We start with a preliminary technical result.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set. If $2 \leq t < r < 2N/(N-1)$, then there exist a constant $C = C(r, t, N, \lambda_1) > 0$ and a number $0 < \theta_1 < 1$ such that*

$$\nu(r, \Omega) \geq C (\nu(t, \Omega))^{\theta_1}.$$

If $2 < r < t < 2N/(N-1)$, then there exist a constant $C = C(r, t, N, \lambda_1) > 0$ and a number $0 < \theta_2 < 1$ such that

$$\nu(r, \Omega) \geq C (\nu(t, \Omega))^{\theta_2}.$$

In particular,

$$\lim_{R \rightarrow +\infty} \nu(r, \mathbb{R}^N \setminus \overline{B_R}) = +\infty$$

if

$$\lim_{R \rightarrow +\infty} \nu(t, \mathbb{R}^N \setminus \overline{B_R}) = +\infty.$$

Proof. We write

$$\frac{1}{r} = \frac{1-\theta}{t} + \frac{\theta}{\frac{2N}{N-1}}$$

for some $0 < \theta < 1$. Then $\|u\|_r \leq \|u\|_t^{1-\theta} \|u\|_{\frac{2N}{N-1}}^\theta$ for every $u \in L^{1/2,2}(\Omega)$. The Gagliardo-Nirenberg inequality (see [28, Theorem 2.1] together with Theorem 2.1) yields $\|u\|_r \leq C \|u\|_{L^{1/2,2}}^\theta \|u\|_t^{1-\theta}$ for every $u \in L^{1/2,2}(\Omega)$. We invoke Lemma 2.3 to get

$$\|u\|_r^2 \leq C \|u\|_X^{2\theta} \|u\|_t^{2(1-\theta)}$$

As a consequence,

$$\begin{aligned} \nu(r, \Omega) &= \inf_{u \in L^{1/2,2}(\Omega) \setminus \{0\}} \frac{\|u\|_X^2}{\|u\|_r^2} \geq \frac{1}{C} \inf_{u \in L^{1/2,2}(\Omega) \setminus \{0\}} \frac{\|u\|_X^2}{\|u\|_V^{2\theta} \|u\|_t^{2(1-\theta)}} \\ &\geq \frac{1}{C} \inf_{u \in L^{1/2,2}(\Omega) \setminus \{0\}} \frac{\|u\|_X^{2(1-\theta)}}{\|u\|_t^{2(1-\theta)}} \geq \frac{1}{C} (\nu(t, \Omega))^{1-\theta}. \end{aligned}$$

The second inequality follows by the same token. \square

Proposition 3.2. *If $\lim_{|x| \rightarrow +\infty} V(x) = +\infty$, then (ν) holds true.*

Proof. By Lemma 3.1 it is sufficient to prove the validity of (ν) with $t_0 = 2$. For any $u \in X$ and any $R > 0$,

$$\int_{\mathbb{R}^N \setminus B_R} \left| (-\Delta + m^2)^{\frac{1}{4}} u \right|^2 + \int_{\mathbb{R}^N \setminus B_R} V |u|^2 \geq \int_{\mathbb{R}^N \setminus B_R} V |u|^2 \geq \inf_{\mathbb{R}^N \setminus B_R} V \int_{\mathbb{R}^N \setminus B_R} |u|^2.$$

Therefore $\nu(2, \mathbb{R}^N \setminus B_R) \geq \inf_{\mathbb{R}^N \setminus B_R} V$, and we conclude by letting $R \rightarrow +\infty$. \square

Lemma 3.3. *Let $\{\omega_n\}_n$ be a sequence of open subset of \mathbb{R}^N such that*

$$\lim_{n \rightarrow +\infty} \mathcal{L}^N(\omega_n) = 0.$$

Then, for every $\varrho > 0$ and every $2 \leq t < 2N/(N-1)$, there results

$$\lim_{n \rightarrow +\infty} \sup \left\{ \int_{\omega_n} |u|^t \mid \|u\|_{L^{1/2,2}} \leq \varrho \right\} = 0.$$

Proof. Since

$$\int_{\omega_n} |u|^t \leq \|u\|_{L^{2N/(N-1)}}^t \mathcal{L}^N(\omega_n)^{\frac{(2-t)N+t}{2N}},$$

by the Sobolev embedding we can find a constant $C_1 > 0$ such that

$$\int_{\omega_n} |u|^t \leq C_1 \|u\|_{L^{1/2,2}}^t \mathcal{L}^N(\omega_n)^{\frac{(2-t)N+t}{2N}} \leq C_2 \varrho^t \mathcal{L}^N(\omega_n)^{\frac{(2-t)N+t}{2N}}$$

whenever $\|u\|_{L^{1/2,2}} \leq \varrho$. The conclusion follows immediately. \square

Proposition 3.4. *Condition (ν') implies condition (ν) .*

Proof. For the sake of contradiction, let us suppose that there exist a sequence $R_n \rightarrow +\infty$ of positive numbers and a sequence $\{u_n\}_n$ of functions from X such that

$$\begin{aligned} \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N \setminus B_{R_n}} \left| (-\Delta + m^2)^{\frac{1}{4}} u_n \right|^2 + \int_{\mathbb{R}^N \setminus B_{R_n}} V |u_n|^2 &< +\infty \\ \int_{\mathbb{R}^N \setminus B_{R_n}} |u_n|^2 &= 1. \end{aligned} \quad (3.1)$$

By Lemma 2.3, the sequence $\{u_n\}$ is bounded in $L^{1/2,2}(\mathbb{R}^N)$. For any $M > 0$ we introduce the set

$$\Omega(M, n) = \left\{ x \in \mathbb{R}^N \setminus B_{R_n} \mid V(x) < M \right\},$$

so that $\lim_{n \rightarrow +\infty} \mathcal{L}^N(\Omega(M, n)) = 0$. For some $C > 0$,

$$\begin{aligned} C &\geq \int_{B_{R_n} \setminus \Omega(M, n)} V |u_n|^2 + \int_{\Omega(M, n)} V |u_n|^2 \geq \\ &M \int_{B_{R_n}} |u_n|^2 - (B + M) \int_{\Omega(M, n)} |u_n|^2. \end{aligned} \quad (3.2)$$

By Lemma 3.3 the last term of (3.2) converges to zero as $n \rightarrow +\infty$. Since $M > 0$ is arbitrary and (3.1) holds true, we derive a contradiction. \square

Finally, we prove that our assumption (ν) is logically equivalent to Sirakov's condition (ν'') .

Proposition 3.5. *Condition (ν'') is equivalent to (ν) .*

Proof. Step 1. Assume that (ν) holds. Given any $r > 0$, we know that

$$\nu(t_0, \mathbb{R}^N \setminus B_{R_n}) \leq \nu(t_0, B(x_n, r))$$

where $R_n = \frac{1}{2}|x_n| - r$. Hence also (ν'') holds.

Step 2. On the contrary, assume that (ν) does *not* hold. Hence there are sequences $\{u_n\}_n$ in X and $\{R_n\}_n$ in $(0, +\infty)$ such that $u_n \in L^{1/2,2}(\mathbb{R}^N)$, $\text{supp } u_n \subset \mathbb{R}^N \setminus \overline{B(0, R_n)}$, $\lim_{n \rightarrow +\infty} R_n = +\infty$,

$$\begin{aligned} \int_{\mathbb{R}^N} \left| (-\Delta + m^2)^{\frac{1}{4}} u_n \right|^2 + \int_{\mathbb{R}^N} V |u_n|^2 &\leq C \\ \int_{\mathbb{R}^N} |u_n|^{t_0} &= 1. \end{aligned}$$

As before, the sequence $\{u_n\}_n$ is bounded in $L^{1/2,2}(\mathbb{R}^N)$ and by a Lemma of P.-L. Lions (see [32, Lemma 2.4]) there exist a sequence $\{x_n\}_n$ of points in \mathbb{R}^N , two constants $r_0 > 0$, $C_0 > 0$, such that, up to a subsequence,

$$\int_{B(x_n, r_0)} |u|^{t_0} \geq C_0. \quad (3.3)$$

It follows from these properties that $|x_n| \rightarrow +\infty$. We fix a sequence of smooth cut-off functions φ_n such that $\varphi_n \in C_c^\infty(\mathbb{R}^N)$, $0 \leq \varphi_n \leq 1$

$$\begin{aligned} \varphi(x) &= 1 \quad \text{if } x \in B(x_n, r_0) \\ &= 0 \quad \text{if } x \notin B(x_n, 2r_0). \end{aligned}$$

Let $v_n = \varphi_n u_n \in L^{1/2,2}(B(x_n, 2r_0))$, so that by (3.3)

$$\int_{B(x_n, 2r_0)} |v_n|^{t_0} \geq C_0.$$

It is well known, see [15], that $\|v_n\|_{H^{1/2}} \leq C_1 \|u_n\|_{H^{1/2}}$ for some constant $C_1 > 0$. By Theorem 2.1 and Lemma 2.3, $\|v_n\|_X \leq C_2 \|u_n\|_X$. Thus for any $n \in \mathbb{N}$,

$$\nu(t_0, B(x_n, 2r_0)) \leq \frac{\int_{B(x_n, 2r_0)} \left| (-\Delta + m^2)^{\frac{1}{4}} u_n \right|^2 + \int_{B(x_n, 2r_0)} V |u_n|^2}{\left(\int_{B(x_n, 2r_0)} |u_n|^{t_0} \right)^{2/t_0}} \leq \frac{C_2 \|u_n\|_X^2}{C_0^{2/t_0}} \leq C_3.$$

We have proved that also (ν'') does *not* hold. \square

4 A compact embedding theorem

Theorem 1.5 will be proved by applying the Symmetric Mountain Pass Theorem of Ambrosetti and Rabinowitz [3] to the functional I . The required compactness is recovered by embedding the space X into a weighted Lebesgue space, see Proposition 4.1.

We are ready to prove the main compactness result of this paper.

Proposition 4.1. *Assume that (H1), (H2), (H3), (H4) and (ν) are satisfied. Then X is compactly embedded into $L^t(\mathbb{R}^N)$ for all $2 \leq t \leq \frac{2(N\beta-1)}{\beta(N-1)}$, and compactly embedded into $L^t(\mathbb{R}^N, A^{2r/(2r-1)} d\mathcal{L}^N)$ if $2 \leq t < \frac{2(N\beta-1)}{\beta(N-1)}$.*

Proof. Consider any bounded sequence $\{u_n\}_n$ in X . By reflexivity, we assume without loss of generality that $u_n \rightharpoonup u$ weakly in X as $n \rightarrow +\infty$. Choose a smooth function $\varphi: \mathbb{R}^N \rightarrow [0, 1]$ with the property that $\varphi(x) = 0$ if $|x| \leq R$, while $\varphi(x) = 1$ if $|x| > R+1$. Then

$$\|u_n - u\|_t \leq \|(1 - \varphi)(u_n - u)\|_{L^t(B_{R+1})} + \|\varphi(u_n - u)\|_{L^t(\mathbb{R}^N \setminus B_R)}.$$

The space $L^{1/2,2}(B_{R+1})$ is compactly embedded into $L^t(B_{R+1})$ because $\frac{2(N\beta-1)}{\beta(N-1)} < 2N/(N-1)$, see Theorem 2.1. Hence, up to a subsequence,

$$\lim_{n \rightarrow +\infty} \|(1 - \varphi)(u_n - u)\|_{L^t(B_{R+1})} = 0.$$

But by definition of ν

$$\nu(t, \mathbb{R}^N \setminus \overline{B_R}) \leq \frac{\|\varphi(u_n - u)\|_X^2}{\|\varphi(u_n - u)\|_{L^t(\mathbb{R}^N \setminus B_R)}}.$$

We deduce that, using the fact that $\|\varphi(u_n - u)\|_{H^{1/2}} \leq C\|u_n - u\|_{H^{1/2}}$ and that $L^{1/2,2} = H^{1/2}$,

$$\|\varphi(u_n - u)\|_{L^t(\mathbb{R}^N \setminus B_R)} \leq \frac{C}{\nu(t, \mathbb{R}^N \setminus \overline{B_R})},$$

and we conclude by Lemma 3.1. The second part follows directly from Proposition 2.4 (see in particular (2.3)). \square

5 Existence of critical points

In this section we apply the celebrated Mountain Pass Theorem and the embedding result proved earlier to find a critical point of the functional I defined in (2.1).

Proposition 5.1. *The functional I has the Mountain Pass geometry.*

Proof. Step 1: there exists a number $\rho > 0$ such that $\|u\|_X = \rho$ implies $I(u) > 0$.

Indeed, by (2.5) we find that for some constant $C > 0$

$$I(u) \geq \frac{1}{2}\|u\|_X^2 - C\|u\|_X^{2p}.$$

Hence $I(u) \leq 0$ implies $\|u\|_X \geq C$ for another constant $C > 0$. The claim follows easily.

Step 2: there exists $e \in X$ such that $\|e\|_X > \rho$ and $I(e) < 0$.

Simply, pick $u \in X$ such that $\int_{\mathbb{R}^N} (K * |u|^{2p})|u|^{2p} > 0$ and compute

$$\lim_{t \rightarrow +\infty} I(tu) = \lim_{t \rightarrow +\infty} \frac{t^2}{2}\|u\|_X^2 - \frac{t^{2p}}{2p} \int_{\mathbb{R}^N} (W * |u|^{2p})|u|^{2p} = -\infty.$$

\square

In order to apply the Mountain Pass Theorem, we need to ensure the validity of the Palais-Smale compactness condition. This follows from Proposition 4.1, as we now show.

Proposition 5.2. *If $\{u_n\}_n$ is any sequence from X such that $dI(u_n) = o(1)$ in X^* and $I(u_n) \leq C$ for every $n \in \mathbb{N}$, then $\{u_n\}_n$ converges strongly in X up to a subsequence.*

Proof. First of all, the sequence $\{u_n\}_n$ is bounded in X . Indeed,

$$C + o(1) \geq I(u_n) - \frac{1}{2p} dI(u_n)(u_n) = \left(\frac{1}{2} - \frac{1}{2p} \right) \|u_n\|_X^2.$$

By reflexivity, we assume without loss of generality that $u_n \rightharpoonup u$ weakly in X as $n \rightarrow +\infty$. It is plain that u is a critical point of I . Now Proposition 4.1 implies that — up to a subsequence — $u_n \rightarrow u$ strongly in $L^{\frac{2rp}{2r-1}}(\mathbb{R}^N, |A|^{\frac{2r}{2r-1}} d\mathcal{L}^N)$, so that (2.5) easily implies

$$\lim_{n \rightarrow +\infty} \|u_n\|_X^2 = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} (W * |u_n|^{2p}) A |u_n|^{2p} = \int_{\mathbb{R}^N} (W * |u|^{2p}) A |u|^{2p} = \|u\|_X^2.$$

We conclude that $u_n \rightarrow u$ strongly in X as $n \rightarrow +\infty$, and the proof is complete. \square

Proof of Theorem 1.5. Let us remark that the right-hand side of equation (1.2) is odd with respect to u . By Proposition 5.1 and Proposition 5.2, we can invoke the Symmetric Mountain Pass Theorem [3] and conclude that the functional I possesses infinitely many critical points in X . \square

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Simone Secchi
Dipartimento di Matematica e Applicazioni
Università degli Studi di Milano Bicocca
via Cozzi 55, 20125 Milano, Italy.
Email: Simone.Secchi@unimib.it
URL: <http://www.matapp.unimib.it/~secchi>